MATH 2028 Honours Advanced Calculus II
2023-24 Term 1
Problem Set 8
due on Nov 24, 2023 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Notations: All curves, surfaces and vector fields are inside $\mathbb{R}^3$. We will use U to denote an open subset of $\mathbb{R}^3$.

Problems to hand in

1. Prove that
   (a) $\nabla \times (\nabla f) = 0$ for any C^2 function $f : U \to \mathbb{R}$;
   (b) $\nabla \cdot (\nabla \times F) = 0$ for any C^2 vector field $F : U \to \mathbb{R}^3$.
2. Compute the flux $\int_S (\nabla \times F) \cdot \vec{n}\, d\sigma$ where
   (a) $F(x, y, z) = (x^2 + y, yz, x - z^2)$ and S is the triangle defined by the plane $2x + y + 2z = 2$ inside the first octant, oriented by the unit normal pointing away from the origin.
   (b) $F(x, y, z) = (x, y, 0)$ and S is the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 4$, oriented by the upward pointing normal.
3. Let $F(x, y, z) = (ye^z, xe^z, xye^z)$ and C be a simple closed curve which is the boundary of a surface S. Show that $\int_C F \cdot\, d\vec{r} = 0$.
4. Find $\iint_S F \cdot \vec{n}\, d\sigma$ where
   (a) $F(x, y, z) = (2x, y^2, z^2)$ and S is the unit sphere centered at the origin, oriented by the outward unit normal;
   (b) $F(x, y, z) = (x + y, y + z, x + z)$ and S is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$, oriented by the outward unit normal.

Suggested Exercises

1. Compute the curl and divergence of the following vector fields:
   (a) $F(x, y, z) = (x^2, xyz, yz^2)$
   (b) $F(x, y, z) = (y \log x, x \log y, xy \log z)$
   (c) $F(x, y, z) = (x^2, \sin xy, e^x yz)$
   (d) $F(x, y, z) = (e^{xy} \sin z, e^{xz} \sin y, e^{yz} \cos x)$
2. A function $f : U \to \mathbb{R}$ is said to be harmonic if $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.

(a) Prove that the functions $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ and $f(x,y,z) = x^2 - y^2 + 2z$ are harmonic on their maximal domain of definition.

(b) Show that $\nabla \cdot (\nabla f) = 0$ if f is harmonic.

3. Let $F(x,y,z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ satisfies $\nabla \cdot F = 0$ and $\nabla \times F = 0$ on $\mathbb{R}^3 \setminus \{0\}$.

4. Calculate the integral $\iint_S (\nabla \times F) \cdot \vec{n}\, d\sigma$ for the vector field $F(x,y,z) = (-y, x^2, z^3)$ and the surface S given by $x^2 + y^2 + z^2 = 1$ with $-1/2 \le z \le 1$.

5. Prove the following identities:

   (a) $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$ for any vector fields F, G.

   (b) $\nabla \cdot (\nabla f \times \nabla g) = 0$ for any functions f, g.

6. Verify Stokes theorem for

   (a) $F(x,y,z) = (z, x, y)$ and S defined by $z = 4 - x^2 - y^2$ and $z \ge 0$;

   (b) $F(x,y,z) = (x, z, -y)$ and S is the portion of the sphere of radius 2 centered at the origin with $y \ge 0$;

   (c) $F(x,y,z) = (y + x, x + z, z^2)$ and S is the portion of the cone $z^2 = x^2 + y^2$ with $0 \le z \le 1$.

7. Compute the flux $\int_S (\nabla \times F) \cdot \vec{n}\, d\sigma$ using Stokes theorem where

   (a) $F(x,y,z) = (y, z, x)$ and S is the triangle with vertices at $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, oriented by the unit normal pointing away from the origin;

   (b) $F(x,y,z) = (x + y, y - z, x + y + z)$ and S is the hemisphere $x^2 + y^2 + z^2 = a^2$ with $z \ge 0$, oriented by the upward pointing normal.

8. Let C be the curve parametrized by

$$\gamma(t) = (\cos t, \sin t, \sin t) \qquad \text{where } t \in [0, 2\pi].$$

   Compute the line integral

$$\int_C z\, dx + 2x\, dy + y^2\, dz$$

   (a) directly from the definition of line integrals; and (b) using Stokes Theorem.

9. Let C be a closed curve which is the boundary of a surface S. Prove that

   (a) $\int_C f \nabla g \cdot\, d\vec{r} = \iint_S (\nabla f \times \nabla g) \cdot \vec{n}\, d\sigma$;

   (b) $\int_C (f \nabla g + g \nabla f) \cdot\, d\vec{r} = 0$.

10. Compute $\iint_S F \cdot \vec{n}\, d\sigma$ for the vector field $F(x,y,z) = (xy, y^2, y^2)$ over the unit cube $S = [0,1] \times [0,1] \times [0,1]$, oriented by the outward normal.

11. Find $\iint_S F \cdot \vec{n}\, d\sigma$ where

   (a) $F(x,y,z) = (x^3, y^3, z^3)$ and S is the unit sphere centered at the origin, oriented by the outward unit normal;

   (b) $F(x,y,z) = (x + y, y + z, x + z)$ and S is the paraboloid $z = 4 - x^2 - y^2$ oriented by the upward unit normal;

(c) $F(x, y, z) = (2x, 3y, z)$ and S is the closed surface consisting of the cylinder $x^2 + y^2 = 4$ and the planes $z = 1$, $z = 3$, oriented by the outward unit normal;

12. Suppose Ω is the interior of a closed surface S. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be C^2 functions. Prove the following *Green's identities*:

    (a) $\iint_S (f\nabla g) \cdot \vec{n}\, d\sigma = \iiint_\Omega (f\Delta g + \nabla f \cdot \nabla g)\, dV$;

    (b) $\iint_S (f\nabla g - g\nabla f) \cdot \vec{v}\, d\sigma = \iiint_\Omega (f\Delta g - g\Delta f)\, dV$;

    Here, $\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

13. Let $\Omega \subset \mathbb{R}^3$ be a bounded open subset with boundary $\partial\Omega = S$ which is a closed surface, oriented by the outward unit normal $\vec{n}$. Let $F(x, y, z) = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$. Assume that $0 \notin S$.

    (a) Suppose that $0 \notin \Omega$. Show that
    $$\iint_S F \cdot \vec{n}\, d\sigma = 0.$$

    (a) Suppose that $0 \in \Omega$. Show that
    $$\iint_S F \cdot \vec{n}\, d\sigma = 4\pi.$$

Challenging Exercises

1. Let $F : U \to \mathbb{R}^3$ be a C^1 vector field defined on an open subset $U \subset \mathbb{R}^3$. Fix $p \in U$. Denote $B_r(p)$ be the closed ball of radius $r > 0$ centered at p and $S_r(p) = \partial B_r(p)$ be the sphere of radius $r > 0$ centered at p, with outward pointing unit normal $\vec{n}$. Prove that
$$(\nabla \cdot F)(p) = \lim_{r\to 0} \frac{1}{\mathrm{Vol}(B_r(p))} \iint_{S_r(p)} F \cdot \vec{n}\, d\sigma.$$

2. Let $S \subset \mathbb{R}^3$ be a surface and $F : U \to \mathbb{R}^3$ be a C^1 vector field defined on an open set $U \subset \mathbb{R}^3$ containing S. Fix $p \in S$. Denote $D_r(p) := \{x \in S \mid |x-p| \le r\}$ and $C_r(p) = \{x \in S \mid |x-p| = r\}$. Suppose S is oriented by the unit normal $\vec{n}$ and so is $C_r(p)$ as the boundary of $D_r(p)$ (which you can assume to be C^1). Prove that
$$(\nabla \times F)(p) \cdot \vec{n}(p) = \lim_{r\to 0} \frac{1}{\mathrm{Area}(D_r(p))} \int_{C_r(p)} F \cdot d\vec{r}.$$